J. Symbolic Computation (2000) **29**, 393–418
doi: 10.1006/jsco.1999.0305
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An Algorithm for Computing a New Normal Form for Dynamical Systems

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We propose in this paper a new normal form for dynamical systems or vector fields which improves the classical normal forms in the sense that it is a further reduction of the classical normal forms. We give an algorithm for an effective computation of these normal forms. Our approach is rational in the sense that if the coefficients of the system are in a field K (which, in practice, is $\mathbf{Q}, \mathbf{R}$), so is the normal form and all computations are done in K. As a particular case, if the matrix of the linear part is a companion matrix then we reduce the dynamical system to a single differential equation. Our method is applicable for both the nilpotent and the non-nilpotent cases. We have implemented our algorithm in Maple V and obtained many examples of the further reduced normal forms up to some finite order.

1. Introduction and Notations

Let K be a commutative field of characteristic zero. We use $K[[x]]$ to denote the ring of formal power series in n variables $x = (x_1, \dots, x_n)$ with coefficients in K. We consider the Poincaré–Dulac normal form problem for an autonomous dynamical system (or the associated vector field)

$$\dot{x} = \frac{dx}{dt} = F(x) \qquad \text{or} \qquad D = f_1\partial_{x_1} + \cdots + f_n\partial_{x_n} \tag{1.1}$$

where $F(x) = (f_1(x), \dots, f_n(x))^t$ is a vector whose components are formal power series without constant terms, i.e. $F(0) = 0$ (0 is a singularity for the dynamical system). One writes $F(x) = \sum_{k\geq 1} F^k(x)$ where $F^k(x)$ is a vector of homogeneous polynomials of degree k. The linear part of the system is $F^1(x) = Ax$, $A \in \mathcal{M}(n, n)$, where $\mathcal{M}(k, m)$ denotes the vector space of $k \times m$ matrices with entries in K.

Let $k \geq 2$. Consider a formal transformation (a near identity change of coordinates) of the form

$$x = y + \varphi^k(y) \tag{1.2}$$

where $\varphi^k(y) \in H_k^n$, H_k denotes the space of homogeneous polynomials of degree k. We have as formal power series

$$(I + \partial_y\varphi^k(y))^{-1} = I - \partial_y\varphi^k(y) + O(\|y\|^{2k-2})$$

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where $\partial_y \varphi^k$ is the Jacobian matrix of φ^k with respect to y and $O(\|y\|^N)$ represents terms of order $\geq N$. Substituting (1.2) into (1.1) we obtain

$$\dot{y} = Ay + \cdots + F^{k-1}(y) + \{F^k(y) - [\partial_y \varphi^k(y)Ay - A\varphi^k(y)]\} + O(\|y\|^{k+1}). \tag{1.3}$$

We introduce a linear operator $L_A^k : H_k^n \longrightarrow H_k^n$ defined by

$$L_A^k(\varphi^k)(y) = [\partial_y \varphi^k(y)]Ay - A\varphi^k(y), \qquad \varphi^k \in H_k^n.$$

Let $\mathcal{R}^k$ be the range of L_A^k in H_k^n and $\mathcal{C}^k$ be any supplementary subspace to $\mathcal{R}^k$ in H_k^n. We have the following decomposition:

$$H_k^n = \mathcal{R}^k \oplus \mathcal{C}^k. \tag{1.4}$$

The fundamental idea of the classical normal form theory is in the following theorem (see, for example, Takens, 1974; Chow and Hale, 1982; Arnold, 1983; Chow, Li and Wang, 1994).

Theorem 1.1. (Takens, 1974) *Consider a dynamical system of the form* (1.1). *Let notations be as above. Let decomposition* (1.4) *be given for* $k = 2, \ldots, N$. *Then there exists a sequence of near identity transformations* $x = y + \varphi^k(y)$ *where* $\varphi^k(y) \in H_k^n$ *such that dynamical system* (1.1) *is transformed into*

$$\dot{y} = Ay + G^2(y) + \cdots + G^N(y) + O(\|y\|^{N+1})$$

where $G^k \in \mathcal{C}^k$ *for* $k = 2, \ldots, N$.

Since our aim in this paper is to compute normal forms up to some finite order, to simplify notations we shall write the normal form of order N in the form

$$\dot{y} = Ay + G^2(y) + \cdots + G^N(y)$$

by ignoring higher order terms.

The aim of the normal form is to determine a change of coordinates such that the new system is as simple as possible. The problems in normal form theory can be stated in different ways:

(1) Given a dynamical system of the form (1.1), how can we compute one of its normal forms?
(2) Given a matrix A, what are the "forms" of the normal forms of all dynamical systems whose linear part is Ax?
(3) Are the normal forms obtained optimal (or unique)? That is to say, no more reduction is possible. In this case the number of parameters remaining in the normal form is minimal.
(4) If the coefficients of the system are in a field K (for example $K = \mathbf{Q}$) can we find a rational normal form? That is to say, the coefficients of the normal form are in K and all the intermediate computations are done in K. This is the rational normal form problem.

Many systematic procedures for constructing normal forms have been given previously. A method of Lie brackets is given in Chow and Hale (1982), Takens (1974) and Ushiki (1984), a method by considering an inner product in the space of homogeneous polynomials is given in Elphick *et al.* (1987) and Ashkenazi and Chow (1988), a method by direct computations is given in Bruno (1979) and Chen and Della Dora (1999b), a method by use of Carleman linearization is given in Tsiligiannis and Lyberatos (1989) and Chen and Della Dora (1999a). The nilpotent case (A is a nilpotent matrix) is treated in Cushman and Sanders (1990) by use of representation theory of $\mathrm{sl}_2(\mathbf{R})$, and in Chen *et al.* (1991) by use of Carleman linearization. The Carleman linearization, introduced in Carleman (1932), has been used in the study of the normal form theory for dynamical systems in Steeb and Wilhelm (1980), Tsiligiannis and Lyberatos (1989), Chen and Della Dora (1999a) and Chen *et al.* (1991).

Most of the classical methods are concerned with problem 2. In this case one usually supposes that the system is in normal form of order $k-1$ and looks for a normal form of order k. One is not concerned with the computation of the diffeomorphism that realizes the normalization nor the changes of terms of order strictly greater than k. However, for problem 1 one needs to compute the diffeomorphism and take account of the changes for higher order terms.

It is known that further reduction is possible for the classical normal forms. A first study in this direction is given in Ushiki (1984) by using the method of Lie brackets. In Baider and Sanders (1992) further reduction has been given in a more general context, that is the graded Lie algebra. Their work specified nilpotent vector fields in dimension 2 into three categories and they have given unique normal forms for the first two categories. Unique normal forms are also given in Baider (1989) and Baider and Churchill (1988) for some cases. In Kokubu *et al.* (1996), the linear grading function method is used to give further reduction in a special case of nilpotent vector fields in dimension 2 for the third category. An algorithm is given in Chen and Della Dora (1999b) for dynamical systems in dimension 2 and 3, which leads to unique normal forms up to some finite order with respect to near identity changes of coordinates. We shall give an algorithm for the general case.

Classical methods use the Jordan canonical form of the leading matrix A. As it is well known, the computation of eigenvalues and the Jordan canonical forms is very difficult in computer algebra systems. Due to this fact these methods are not effective for an implementation in a computer algebra system.

Using the Carleman linearization procedure and a Frobenius basis in H_k we introduced in Chen and Della Dora (1999a) a rational method for the normal form of any dynamical system. This normal form is an improvement of the classical normal form. We proposed an algorithm for the computation of both the classical and the improved normal forms. However, the manipulation of the Frobenius bases is complicated. We shall now propose another choice for the normal form which does not use the Frobenius basis. We use, as in Chen and Della Dora (1999a), the Frobenius canonical form of the linear part instead of the Jordan canonical form in the classical methods. Thus we do not need to compute the Jordan canonical form of A or its eigenvalues and all computations are done in the field K. Our method is applicable for both the non-nilpotent and the nilpotent cases. We will provide many examples of normal forms. These examples of normal forms are central to the work and contribute significantly to its length.

In Section 2 the Carleman linearization process which is used in this paper is given. In Section 3 we recall the classical normal form theorem in our context and the further reduced normal form of Chen and Della Dora (1999b). In Section 4 we give a new rational normal form which is an improvement of the classical normal form. We have implemented our algorithm in Maple V. In Section 5 we shall consider examples of normal forms compared with the classical normal forms.

We give here an example to show the type of normal forms obtained by our algorithm and the discussions which may occur.

Let $A = \begin{pmatrix} 0 & 1 \\ -1/2 & 0 \end{pmatrix}$. One notes that its eigenvalues are $\pm i\sqrt{2}/2$. One can choose a classical normal form up to any order (see Chen and Della Dora, 1999b):

$$\begin{aligned} \dot{x}_1 &= x_2 + \sum_{j\geq 1} \alpha_j x_1^{2j+1}, \\ \dot{x}_2 &= -\frac{1}{2}x_1 + \sum_{j\geq 1} \beta_j x_1^{2j+1}. \end{aligned}$$

If $\alpha_1 \neq 0$, then we have the following rational normal form up to order 11:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{1}{2}x_1 + \mu_1 x_1^3 + \mu_2 x_1^2 x_2 + \mu_3 x_1^4 x_2. \end{aligned}$$

If $\alpha_1 = 0$ and $\beta_1 \neq 0$, then we have the following rational normal form up to order 11:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{1}{2}x_1 + \mu_1 x_1^3 + \mu_2 x_1^4 x_2 + \mu_3 x_1^6 x_2 + \mu_4 x_1^8 x_2. \end{aligned}$$

Here the μ_j's are parameters depending rationally on the α_j and β_j. Moreover, these normal forms are optimal (or unique) up to the given order with respect to near identity diffeomorphisms.

2. Carleman Linearization

2.1. Carleman linearization of derivations

Let $n \geq 1$ and $x = (x_1, x_2, \ldots, x_n)^t$. For all integer $k \geq 0$, $H_k(x)$ or simply H_k will denote the space of homogeneous polynomials of degree k in n variables $x_1, x_2, \ldots, x_n$. In $H_k(x)$ we choose the canonical basis x^q, with an n-tuple index $q = (q_1, \ldots, q_n)$, $q_i \in \mathbf{N}$ and $|q| = q_1 + \cdots + q_n = k$. We choose the lexicographical order induced by the order $x_1 < x_2 < \cdots < x_n$ on the set of monomials $\{x^q : |q| = k\}$. We denote this basis by

$$e_1^k = x_1^k,\ e_2^k = x_1^{k-1}x_2, \ldots, e_{d_k}^k = x_n^k$$

and $m_k = (e_1^k, \ldots, e_{d_k}^k)^t$ where $d_k = \begin{pmatrix} n+k-1 \\ k \end{pmatrix}$ is the dimension of $H_k(x)$. One has in particular $d_0 = 1, m_0 = 1, d_1 = n, m_1 = (x_1, \ldots, x_n)^t$. Thus any element of $H_k(x)$ can be written as

$$P(x) = \sum_{|q|=k} \alpha_q x^q = (\beta_1, \ldots, \beta_{d_k})m_k$$

where $\alpha_q \in K$ and the β's are rearrangement of the α's according to the basis e_i^k. It is clear now that any element of $K[[x]]$ can be represented by $f = \sum_{k=0}^{+\infty} a_k m_k$ with $a_k \in K^{d_k}$. In this notation $S \in K[[x]]^n$ can be written as

$$S(x) = \sum_{k=0}^{+\infty} D_{1k} m_k$$

where $D_{1k} \in \mathcal{M}(n, d_k)$.

The basic idea of Carleman linearization is to associate to dynamical system (1.1) a derivation D acting on $K[[x]]$. This derivation is defined as the directional derivation in the direction of the vector field defined by F in (1.1): $D\phi = \langle F, \bigtriangledown\phi\rangle$ (where $\bigtriangledown\phi$ is the gradient of ϕ). In particular, $Dx_i = f_i(x)$ where $f_i(x)$ is the ith component of $F(x)$. Thus

$$Dm_1 = \begin{pmatrix} Dx_1 \\ \vdots \\ Dx_n \end{pmatrix} = \sum_{i=1}^{+\infty} D_{1i} m_i \tag{2.1}$$

where D_{1i} are $n \times d_i$ matrices, $F^k(x) = D_{1k}m_k$ is the homogeneous part of degree k of the system and $D_{11} = A$.

We extend the action of D to the elements of the basis m_i of $H_i(x)$ for $i \geq 2$. For example, the action of D on the component x_1x_2 of m_2 is $D(x_1x_2) = x_2Dx_1 + x_1Dx_2$, where Dx_1 and Dx_2 are known by (2.1). Similarly,

$$D(x_1^2) = 2x_1D(x_1) \qquad \text{and} \qquad D(x_2^2) = 2x_2D(x_2).$$

For $n = 2$ we have

$$\begin{aligned} Dm_2 = D\begin{pmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix} &= \begin{pmatrix} 2x_1 & 0 \\ x_2 & x_1 \\ 0 & 2x_2 \end{pmatrix} \begin{pmatrix} Dx_1 \\ Dx_2 \end{pmatrix} \\ &= \begin{pmatrix} 2x_1 & 0 \\ x_2 & x_1 \\ 0 & 2x_2 \end{pmatrix} \sum_{j\geq 1} D_{1j}m_j = \sum_{k\geq 2} D_{2k}m_k. \end{aligned}$$

By recurrence, for $i \geq 2$, we can calculate:

$$Dm_i = \sum_{j\geq i} D_{ij}m_j.$$

Let $H^\infty = H_1 \oplus H_2 \oplus \cdots$. Then D is a linear map from H^∞ to H^∞. It has an infinite matrix representation $T_m(D)$ in the basis

$$m = (m_1^t, m_2^t, m_3^t, \ldots)^t.$$

We can write

$$Dm = T_m(D)m \tag{2.2}$$

where $T_m(D)$ is an infinite upper block triangular matrix of the form:

$$\begin{pmatrix} D_{11} & D_{12} & D_{13} & \cdots \\ & D_{22} & D_{23} & \cdots \\ & & D_{33} & \cdots \\ & & & \ddots \end{pmatrix}$$

with $D_{ij} (i \leq j)$ a $d_i \times d_j$ matrix.

Remark 2.1. The matrix $T_m(D)$ is uniquely determined by the matrices $D_{1k}(k \geq 1)$ in (2.1). Hence we will also denote this matrix by $\mathrm{Der}_m(D_{11}, D_{12}, \ldots)$. And we will denote by

$$\mathrm{Der}_m^k(D_{11}, \ldots, D_{1k}) \qquad \text{or} \qquad T_m^{(k)}(D),$$

the following truncated matrix:

$$T_m^{(k)}(D) = \begin{pmatrix} D_{11} & D_{12} & D_{13} & \cdots & D_{1k} \\ & D_{22} & D_{23} & \cdots & D_{2k} \\ & & D_{33} & \cdots & D_{3k} \\ & & & \ddots & \vdots \\ & & & & D_{kk} \end{pmatrix} \tag{2.3}$$

i.e. the truncation of the matrix at the order k.

EXAMPLE 2.1. Consider a derivation D associated to the following dynamical system

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \alpha_{20}x_1^2 + \alpha_{11}x_1x_2 + \alpha_{02}x_2^2,$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \beta_{20}x_1^2 + \beta_{11}x_1x_2 + \beta_{02}x_2^2.$$

One then has $T_m^{(3)}(D) =$

$$\begin{bmatrix} a_{11} & a_{12} & \alpha_{20} & \alpha_{11} & \alpha_{02} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & \beta_{20} & \beta_{11} & \beta_{02} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\,a_{11} & 2\,a_{12} & 0 & 2\,\alpha_{20} & 2\,\alpha_{11} & 2\alpha_{02} & 0 \\ 0 & 0 & a_{21} & a_{11}+a_{22} & a_{12} & \beta_{20} & \alpha_{20}+\beta_{11} & \alpha_{11}+\beta_{02} & \alpha_{02} \\ 0 & 0 & 0 & 2\,a_{21} & 2\,a_{22} & 0 & 2\beta_{20} & 2\,\beta_{11} & 2\beta_{02} \\ 0 & 0 & 0 & 0 & 0 & 3\,a_{11} & 3\,a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{21} & 2a_{11}+a_{22} & 2a_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\,a_{21} & a_{11}+2\,a_{22} & a_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\,a_{21} & 3\,a_{22} \end{bmatrix}.$$

2.2. Carleman linearization of diffeomorphisms

Let $\mathcal{G}_n$ be the group of formal automorphisms tangent to the identity of $K[[x]]^n$ and let

$$\varphi = (\varphi_1(x), \ldots, \varphi_n(x))^t \in \mathcal{G}_n.$$

We now introduce the matrix representation of φ. Let us write

$$\varphi(m_1) = m_1 + \sum_{i\geq 2} T_{1i}m_i$$

where T_{1i} is an element of $\mathcal{M}(n, d_i)$. We can extend these representations to other basis vectors:

$$\varphi(m_j) = m_j + \sum_{k>j} T_{jk}m_k = \sum_{k\geq j} T_{jk}m_k$$

using the properties of φ. For instance, if $n = 2$, then one has

$$\varphi(m_2) = \varphi \begin{pmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{pmatrix} = \begin{pmatrix} \varphi_1(x)^2 \\ \varphi_1(x)\varphi_2(x) \\ \varphi_2(x)^2 \end{pmatrix} = m_2 + \cdots.$$

Then φ, as a linear map on H^∞ still denoted by φ, can be represented by the following infinite upper block triangular matrix:

$$T_m(\varphi) = \begin{pmatrix} I_1 & T_{12} & T_{13} & \cdots \\ & I_2 & T_{23} & \cdots \\ & & I_3 & \cdots \\ & & & \ddots \end{pmatrix} \tag{2.4}$$

where in the diagonal I_i is the identity matrix of order d_i and T_{ij} is a $d_i \times d_j$ matrix. We can write

$$\varphi(m) = T_m(\varphi)m.$$

Remark 2.2. Matrix representation (2.4) depends only on the matrices: I_1, T_{12}, T_{13}, $\ldots$. Therefore we can also use the notation $\mathrm{Diff}_m(I_1, T_{12}, T_{13}, \ldots)$ for this matrix and we denote by $\mathrm{Diff}^k_m(I_1, T_{12}, \ldots, T_{1k})$ or $T^{(k)}_m(\varphi)$ the corresponding matrix truncated at the order k as for the derivation above.

Example 2.2. Let $\varphi = (\varphi_1(x), \varphi_2(x))^t$ with

$$\begin{aligned} \varphi_1(x) &= x_1 + a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2, \\ \varphi_2(x) &= x_2 + b_{20}x_1^2 + b_{11}x_1x_2 + b_{02}x_2^2. \end{aligned}$$

Then

$$T^{(3)}_m(\varphi) = \begin{bmatrix} 1 & 0 & a_{20} & a_{11} & a_{02} & 0 & 0 & 0 & 0 \\ 0 & 1 & b_{20} & b_{11} & b_{02} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2\,a_{20} & 2\,a_{11} & 2\,a_{02} & 0 \\ 0 & 0 & 0 & 1 & 0 & b_{20} & b_{11}+a_{20} & b_{02}+a_{11} & a_{02} \\ 0 & 0 & 0 & 0 & 1 & 0 & 2\,b_{20} & 2\,b_{11} & 2\,b_{02} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

One can represent the action of D and φ by the following diagram:

$$\begin{tikzcd} H^\infty \arrow[r, "D"] \arrow[d, "\varphi"'] & H^\infty \arrow[d, "\varphi"'] \\ H^\infty \arrow[r, "D"] & H^\infty \end{tikzcd} \tag{2.5}$$

We are going to construct a formal diffeomorphism φ such that $\varphi \circ D \circ \varphi^{-1}$ is as simple as possible (in a sense to be specified later on).

We will use φ as a change of variables for dynamical system (1.1). If $m'_i = \varphi(m_i)$, then $m' = ({m'_1}^t, {m'_2}^t, \ldots)^t$ is a basis in H^∞ and $m' = T_m(\varphi)m$. One then has

$$Dm' = T_m(\varphi)Dm = T_m(\varphi)T_m(D)m = T_m(\varphi)T_m(D)T_m(\varphi)^{-1}m'.$$

Hence the new matrix of D in the basis m' is

$$T_{m'}(D) = T_m(\varphi)T_m(D)T_m(\varphi)^{-1}.$$

Lemma 2.1. *Let $\psi, \varphi \in \mathcal{G}_n$, then $\psi \circ \varphi \in \mathcal{G}_n$ and $\varphi^{-1} \in \mathcal{G}_n$. Let $m' = \varphi(m)$, then we have*

$$T_m(\psi \circ \varphi) = T_{m'}(\psi)T_m(\varphi) \qquad \text{and} \qquad T_{m'}(\varphi^{-1}) = T_m(\varphi)^{-1}.$$

Proof. In fact, if $T_{m'}(\psi) = (U_{ij})$ and $T_m(\varphi) = (T_{ij})$, then for any $j \geq 1$,

$$\psi \circ \varphi(m_j) = \sum_{i \geq j} U_{ji}\varphi(m_i) = \sum_{i \geq j}\sum_{k \geq i} U_{ji}T_{ik}m_k = \sum_{k \geq j}\sum_{j \leq i \leq k} U_{ji}T_{ik}m_k = \sum_{k \geq j} V_{jk}m_k$$

where

$$V_{jk} = \sum_{j \leq i \leq k} U_{ji}T_{ik}.$$

Hence $T_m(\psi \circ \varphi) = T_{m'}(\psi)T_m(\varphi)$. It follows immediately that $T_{m'}(\varphi^{-1}) = T_m(\varphi)^{-1}$.□

In practice one needs to compute the inverse of the matrix $T_m^{(k)}(\varphi)$. This can be done easily since the matrix $T_m^{(k)}(\varphi)$ is an upper block triangular matrix. In fact, if one writes $T_m^{(k)}(\varphi)^{-1} = (U_{ij}), 1 \leq i, j \leq k$, then $U_{ii} = I_i$, $U_{ij} = 0$ if $i > j$ and $U_{ij} (i < j)$ is a $d_i \times d_j$ matrix which can be computed recursively by the relations:

$$U_{ij} = -\, U_{ii}T_{ij} - \cdots - U_{i,j-1}T_{j-1,j}$$

for i from k to 1 and for j from $i+1$ to k.

The preceding diagram (2.5) can be translated now to the following diagram:

$$\begin{array}{ccc} H^\infty & \xrightarrow{T_m(D)} & H^\infty \\ {\scriptstyle T_m(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle T_m(\varphi)} \\ H^\infty & \xrightarrow{T_{m'}(D)} & H^\infty \end{array}$$

Our problem is then to find $T_m(\varphi)$ such that $T_m(\varphi)T_m(D)T_m(\varphi)^{-1}$ is as simple as possible (i.e. contains as many zeros as possible). In practice, if we look for a normal form of order N we have to construct $T_m^{(N)}(\varphi)$ such that $T_m^{(N)}(\varphi)T_m^{(N)}(D)T_m^{(N)}(\varphi)^{-1}$ is as simple as possible.

3. Fundamental Theorems of Normal Form Theory

We recall in this section the fundamental theorem of the classical normal form in our context and the further reduced normal form of Chen and Della Dora (1999b). We shall give another choice of the normal form and its improvement in the next section.

3.1. Takens' theorem in matrix form

Our approach of the problem uses the infinite dimensional matrix formalism intensively. We want to build $T_m(\varphi)$ such that $T_m(\varphi) \cdot T_m(D) \cdot T_m(\varphi)^{-1}$ is in a simpler form.

Let N be an integer ≥ 2. Consider the derivation D associated to (1.1) defined in the above section. As stated in Section 1 we want to build a normal form of order N. For this purpose we shall work with the truncated representation of D, i.e. $\mathrm{Der}_m^N(D_{11}, \ldots, D_{1N})$ or $T_m^{(N)}(D)$. The previous normal form problem is reduced to eliminating as many as possible non-zero elements in the matrices $D_{1k}(2 \leq k \leq N)$.

The general theory of normal form consists of fixing a normal form of order $\leq k-1$ and looking for the normal form of order k. So we look for a formal diffeomorphism $\varphi(x) = x + \varphi^k(x)$ where φ^k is a vector of homogeneous polynomials of degree k, i.e.

$$m_1' = \varphi(m_1) = m_1 + E_k m_k.$$

Then one can compute its matrix representation

$$T_m(\varphi) = \mathrm{Diff}_m(I, 0, \ldots, 0, E_k, 0, \ldots),$$

as in Section 2. In the matrix $T_m(\varphi)\cdot T_m(D)\cdot T_m(\varphi)^{-1}$ the first $\sum_{i=1}^{k-1} d_i$ rows and columns are unchanged. Since we are building a normal form of order k we do not take care of terms of degrees $> k$ at the moment. So for simplicity we work with the following matrix instead of $\mathrm{Der}_m^N(D_{11}, \ldots, D_{1N})$:

$$M = \begin{pmatrix} D_{11} & D_{1k} \\ 0 & D_{kk} \end{pmatrix}.$$

Denote by $\mathrm{Elem}(E_k)$ the elementary matrix of the following form

$$\mathrm{Elem}(E_k) = \begin{pmatrix} I_1 & E_k \\ 0 & I_k \end{pmatrix},$$

where I_j is the identity matrix of order d_j for any $j \geq 1$ and E_k is an $n \times d_k$ matrix. It is clear that

$$\mathrm{Elem}(E_k)^{-1} = \mathrm{Elem}(-E_k)$$

and $\mathrm{Elem}(E_k) \cdot \mathrm{Elem}(E_k') = \mathrm{Elem}(E_k + E_k')$. Our problem is to find a matrix E_k such that in the resulting matrix

$$\mathrm{Elem}(E_k) \cdot M \cdot \mathrm{Elem}(-E_k) = \begin{pmatrix} D_{11} & D_{1k}' \\ 0 & D_{kk} \end{pmatrix}$$

the matrix D_{1k}' contains as many zeros as possible. Define a linear map (the homological operator)

$$L_k : \mathcal{M}(n, d_k) \longrightarrow \mathcal{M}(n, d_k)$$

by $L_k(E_k) = D_{11}E_k - E_k D_{kk}$. We then have

$$D_{1k}' = D_{1k} - L_k(E_k).$$

Let $\mathcal{R}^k$ be the range of L_k in $\mathcal{M}(n, d_k)$ and $\mathcal{C}^k$ be any of its supplementary subspace in $\mathcal{M}(n, d_k)$. We have the following decomposition:

$$\mathcal{M}(n, d_k) = \mathcal{R}^k \oplus \mathcal{C}^k. \tag{3.1}$$

In our context the classical normal form theorem of Takens can be written as:

Theorem 3.1. *Consider a dynamical system of the form* (1.1) *with the truncated representation* $\mathrm{Der}_m^N(D_{11}, \ldots, D_{NN})$. *Let notations be as above. Suppose that we have a* *decomposition of the form* (3.1) *for any* $2 \leq k \leq N$. *Then there exists a formal diffeomorphism* $\varphi \in \mathcal{G}_n$ *with*

$$T_m^{(N)}(\varphi) = \mathrm{Diff}_m^N(I_1, T_{12}, \ldots, T_{1N})$$

such that

$$T_m^{(N)}(\varphi \circ D \circ \varphi^{-1}) = \mathrm{Der}_m^N(D_{11}, D'_{12}, \ldots, D'_{1N})$$

where $D'_{1k} \in \mathcal{C}^k$ *for all* $2 \leq k \leq N$.

It is known that this normal form is not unique. It depends on the choice of $\mathcal{C}^k$. It is not unique even with fixed $\mathcal{C}^k$.

3.2. FURTHER REDUCTIONS OF THE TAKENS' NORMAL FORM

We now consider further reduction of the Takens' normal form.

Let ℓ be an integer such that $\dim \mathcal{C}^\ell \geq 1$. We then have $\dim \mathrm{Ker}(L_\ell) = \dim \mathcal{C}^\ell \geq 1$. For any $E'_\ell \in \mathrm{Ker}(L_\ell)$,

$$L_\ell(E_\ell + E'_\ell) = L_\ell(E_\ell).$$

Let $k > \ell$. According to Theorem 3.1 and the previous section, there exists a formal diffeomorphism φ with

$$T_m^{(k)}(\varphi) = \mathrm{Diff}_m^k(I, 0, \ldots, 0, E_\ell + E'_\ell, E_{\ell+1}, \ldots, E_k)$$

such that the transformed system

$$\mathrm{Der}_m^k(D_{11}, \ldots, D'_{1\ell}, \ldots, D'_{1k})$$

is in the normal form of Theorem 3.1, i.e. $D'_{1j} \in \mathcal{C}^j$ for $\ell \leq j \leq k$. One can write

$$D'_{1k} = \tilde{D}_{1k} - \hat{D}_{1k}$$

where $\tilde{D}_{1k}$ and $\hat{D}_{1k}$ belong to $\mathcal{C}^k$ and where $\hat{D}_{1k}$ contains all terms depending on E'_ℓ. Define a non-linear operator

$$N_{\ell,k} : \mathrm{Ker}(L_\ell) \longrightarrow \mathcal{C}^k$$

by $N_{\ell,k}(E'_\ell) = \hat{D}_{1k}$. Let $\mathcal{R}_2^k$ be a subspace contained in the range of $N_{\ell,k}$. Then one can find a supplementary subspace $\mathcal{C}_2^k$ in $\mathcal{C}^k$ such that

$$\mathcal{C}^k = \mathcal{R}_2^k \oplus \mathcal{C}_2^k. \tag{3.2}$$

The following is an improvement of the classical normal form Theorem 3.1 in the present context.

THEOREM 3.2. (CHEN AND DELLA DORA, 1999B) *Consider a dynamical system of the form* (1.1) *with the truncated representation* $\mathrm{Der}_m^N(D_{11}, \ldots, D_{NN})$. *Let notations be as above. Let* ℓ *be an integer such that* $\mathrm{Ker}(L_\ell) \neq \{0\}$. *Suppose that there exist* $E_j (\ell \leq j \leq k)$ *such that there is a non-trivial subspace* $\mathcal{R}_2^k$ *contained in the range of* $N_{\ell,k}$ *with decomposition* (3.2). *Then there exists* $E'_\ell \in \mathrm{Ker}(L_\ell)$, *which implies a formal diffeomorphism* φ *with*

$$T_m^{(k)}(\varphi) = \mathrm{Diff}_m^k(I_1, 0, \ldots, 0, E_\ell + E'_\ell, \ldots, E_k),$$

such that the transformed system is in the form

$$\mathrm{Der}^k_m(D_{11}, \ldots, D'_{1\ell}, \ldots, D'_{1,k-1}, D'_{1k})$$

where $D'_{1j} \in \mathcal{C}^j (\ell \le j \le k-1)$ *and* $D'_{1k} \in \mathcal{C}^k_2$.

Note that Ushiki (1984) and Gaeta (1999) have used $\mathrm{Ker}(L_\ell)$ to give further reduction of higher order terms of the classical normal form in a different way. All the examples given in Gaeta (1999) are with semi-simple linear parts. We shall compare our results with classical methods in Section 5.

Using a Frobenius basis in H_k we have given in Chen and Della Dora (1999a) a choice of $\mathcal{C}^k$ and $\mathcal{C}^k_2$ to obtain a further reduced normal form and an algorithm for an effective computation of both the classical normal form and the further reduced normal form. As in the classical methods, the bases obtained for $\mathcal{C}^k$ are in general composed by homogeneous polynomials (not monomials). In this paper we provide another choice of $\mathcal{C}^k$ and $\mathcal{C}^k_2$ to obtain another further reduced normal form which is easier to compute. The bases obtained are always composed by monomials. In the particular case where the matrix of the linear part is a companion matrix, the dynamical system is reduced to an nth order single non-linear differential equation. Examples are given in Section 5.

4. A New Normal Form and its Further Reductions

The traditional way to handle the problem is to transform the linear part of the dynamical system to its Jordan canonical form. This way of handling the problem introduces both theoretical and practical difficulties that are unsolved by the traditional algorithms. For instance, the computation of the matrix eigenvalues and the well known related problem of recognition of the resonant monomials. Here we start with a linear part reduced to a weak Frobenius canonical form, i.e. a block diagonal matrix with companion matrices in the diagonal. Computation of a Frobenius canonical form can be done easily as implemented in several computer algebra systems (see also Ozello, 1987; Chen, 1989; Gil, 1993; and Storjohann, 1998).

Let N be an integer ≥ 2 and D the derivation defined as above by

$$D(m_1) = \sum_{j=1}^{\infty} D_{1j} m_j. \tag{4.1}$$

Recall that D_{1j} belongs to $\mathcal{M}(n, d_j)$ with coefficients in the field K. We suppose that the linear part D_{11} of the system is in a weak Frobenius canonical form, i.e.

$$D_{11} = \mathrm{diag}(C_1, \ldots, C_r) \tag{4.2}$$

where C_i are companion matrices. For instance

$$C_i = \mathrm{companion}(a_j)_{1\le j\le n_i} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_{n_i} \end{pmatrix}.$$

Remark that we need that, in (4.2), all the matrices C_i are in the companion form. However we do not need that the characteristic polynomial of C_i divides that of C_{i-1} as is needed in the Frobenius canonical form.

We require the following lemma.

Lemma 4.1. *Let* $F_1 = \text{companion}(a_j)_{1\leq j\leq \nu}$ *be a companion matrix of order* ν *with* $\nu > 1$ *and* $a_j \in K$*. Let* F_2 *be any* $d_k \times d_k$ *matrix with entries in* K*. Then for any* $\nu \times d_k$ *matrix* P *with entries in* K *one can compute a* $\nu \times d_k$ *matrix* Q *with coefficients in* K *such that*

$$QF_2 - F_1 Q + P = B \tag{4.3}$$

where B *is a* $\nu \times d_k$ *matrix with coefficients in* K *such that only the last row may be non-zero.*

In many cases one may reduce some or all of the elements in the last row of B *to zero.*

Proof. For $j = 1, \dots, \nu$ we denote by p_j, q_j and b_j the jth row of the matrices P, Q and B, respectively. Let q_1 be given arbitrarily.

For $j = 1, \dots, \nu - 1$, the jth row of matrix equation (4.3) is

$$q_j F_2 - q_{j+1} + p_j = b_j.$$

One can determine q_{j+1} such that $b_j = 0$ for $1 \leq j \leq \nu - 1$. In fact for $1 \leq j \leq \nu - 1$,

$$q_{j+1} = q_j F_2 + p_j.$$

The matrix B is in the desired form.

The νth row of equation (4.3) is

$$q_\nu F_2 - a_1 q_1 - \cdots - a_\nu q_\nu + p_\nu = b_\nu.$$

As $q_2, \dots, q_\nu$ depend linearly on q_1, then in many cases one may compute some elements of q_1 to annul some of the elements of b_ν. $\square$

We have the following normal form theorem.

Theorem 4.1. *Consider a dynamical system of the form* (1.1) *with matrix representation* (2.2)*. Let notations be as in the previous sections. Suppose that the matrix* $D_{11} = A$ *is in form* (4.2)*. For any integer* $k \geq 2$ *one can reduce the dynamical system to a normal form* $\text{Der}(D_{11}, D'_{12}, \dots, D'_{1k}, \dots)$ *where* D'_{1k}*, the homogeneous part of degree* k*, contains non-zero elements only in the rows corresponding to the last rows of each* C_i*.*

Moreover many of these elements can be reduced to zero.

Proof. We prove the theorem by an algorithm which constructs a diffeomorphism $T_m^{(N)}(\varphi)$ such that $T_m^{(N)}(\varphi) T_m^{(N)}(D) T_m^{(N)}(\varphi)^{-1}$ is in the desired normal form.

We suppose that we have obtained a normal form of order $k-1$ and look for a normal form of order k. As in Section 3 we work with the following matrix to simplify notations

$$M = \begin{pmatrix} D_{11} & D_{1k} \\ 0 & D_{kk} \end{pmatrix}.$$

The following gives a clear description of the blocks in the matrix M:

$$M = \begin{pmatrix} C_1 & & & P_1 \\ & \ddots & & \vdots \\ & & C_r & P_r \\ & & & D_{kk} \end{pmatrix}.$$

We apply Lemma 4.1 with $F_1 = C_i$, any block of D_{11}, $F_2 = D_{kk}$ and $P = P_i$ to obtain a matrix that we denote by Q_i. It is clear that what remain in the normal form are the last rows of the matrices B_i, i.e. the row corresponding to the last row of C_i. We form the matrix $\mathrm{Elem}(E_k)$ with

$$E_k = \begin{pmatrix} Q_1 \\ \vdots \\ Q_r \end{pmatrix}.$$

Then

$$\mathrm{Elem}(E_k) \cdot M \cdot \mathrm{Elem}(E_k)^{-1} = \begin{pmatrix} D_{11} & D'_{1k} \\ 0 & D_{kk} \end{pmatrix}$$

where D'_{1k} is in the desired form of Theorem 3.2.

We now return to the matrix representation $\mathrm{Der}_m^{(N)}(D_{11}, \ldots, D_{1N})$.

According to Section 2, one can compute the matrix representation $T_m^{(N)}(\varphi)$ of φ according to $\varphi(m_1) = m_1 + E_k m_k$. One then has

$$T_m^{(N)}(\varphi) = \mathrm{Diff}_m^N(I_1, 0, \ldots, 0, E_k, 0, \ldots, 0).$$

According to the above computations,

$$T_m^{(N)}(\varphi) T_m^{(N)}(D) T_m^{(N)}(\varphi)^{-1} = \mathrm{Der}_m^N(D_{11}, \ldots, D'_{1k}, \ldots, D'_{1N})$$

where D'_{1k} is in the rational normal form described in the theorem.□

One can repeat the above computations to obtain a normal form up to order N.

Corollary 4.1. *Let notations be as in the above theorem, if A is a companion matrix, then the dynamical system can be converted to a normal form up to order N, which is equivalent in an obvious way to a single nth order non-linear differential equation.*

Further reductions. The above algorithm can be used to make further reductions of the normal form.

In Lemma 4.1 there may exist arbitrary elements in Q. Thus there exist elements in E_k undetermined if $\mathcal{C}_k \neq \{0\}$. These undetermined elements are the key tools for simplifying higher order normal form. In fact we continue to compute the normal form of order $k+1$. If in the homogeneous part of degree $k+1$ of the normal form there is a term α depending linearly on an undetermined element of E_k, then one can solve the equation $\alpha = 0$ for this element of E_k. This reduces one more parameter to zero in the normal form of order $k+1$. The same procedure applies to higher order normal forms. This leads to the further reduced normal form as stated in Theorem 3.2. The examples in the next sections will illustrate the type of discussions that can occur during this further reduction step.

5. Examples of Normal Forms

The examples given in this section are computed using Maple V with an implementation of the above algorithm. Our method computes at the same time the formal transformation that realizes the normalization. However we shall just give the normal forms to simplify the presentation. Comparisons with other methods are given for each example. We first provide a general remark which is used in some of the following examples.

Remark 5.1. Let A be in a fixed canonical form. If P is an invertible matrix such that $P^{-1}AP = A$, then the linear transformation $x = Py$ will not change the linear part of the system. However one may use the arbitrary parameters in P to further reduce some higher order terms.

5.1. Dynamical systems of dimension 2

We first study dynamical systems of dimension 2 of the following form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = F(x) = Ax + \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \tag{5.1}$$

with coefficients in a field K, where A is the matrix of the linear part.

Example 5.1. Consider a dynamical system of the form (5.1) with the nilpotent matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{5.2}$$

as its linear part.

Note that the methods of Elphick *et al.* (1987) and Cushman and Sanders (1990) lead to the following normal form of any order N:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 x_2 P_1(x_1) + x_1^2 P_2(x_1) \tag{5.3}$$

where $P_1(x_1)$ and $P_2(x_1)$ are polynomials of degree $N-2$ in x_1. This normal form contains two non-zero parameters in the homogeneous part of any order, the same as in the normal form of Takens (1974).

Ushiki (1984) studied further reductions of the Takens' normal form in this case. He obtained a further reduced normal form of order 4 (see also Chua and Kokubu, 1989). We obtained in Chen and Della Dora (1999b) a further reduced normal form of order 9 by a different method.

To simplify notations we shall apply our algorithm to systems which are in the Takens' normal form (see Takens, 1974 and Chen and Della Dora, 1999b). That is to say we suppose that in system (5.1)

$$f_1 = \sum_{k\geq 2} \alpha_k x_1^k, \qquad f_2 = \sum_{k\geq 2} \beta_k x_1^k.$$

Note first that if

$$P = \begin{pmatrix} u & v \\ 0 & u \end{pmatrix}$$

where $u \neq 0$, then $P^{-1}AP = A$. The linear transformation $x = Py$ will not change the linear part of the system. One may choose appropriate u and v to give further reduction of higher order terms. For example if $\beta_2 \neq 0$, then one can choose $u = 1/\beta_2$ to reduce β_2 to 1.

For $k = 2$, we apply Lemma 4.1 with $F_1 = A$ and

$$F_2 = D_{22} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We obtain a matrix

$$Q = (Q_{ij}) = \begin{bmatrix} 0 & Z_1 & Z_2 \\ \alpha_2 & 0 & Z_1 \end{bmatrix},$$

where $Z_1, Z_2 \in K$ are arbitrary, such that the matrix B in Lemma 4.1 is

$$B = \begin{bmatrix} 0 & 0 & 0 \\ \beta_2 & 2\,\alpha_2 & 0 \end{bmatrix}.$$

In the matrix Q there are two undetermined elements Q_{12} and Q_{13} which are denoted by Z_1 and Z_2. We build the formal diffeomorphism: $\varphi(m_1) = m_1 + Qm_2$. We can then compute its matrix representation and therefore we obtain a rational normal form of order 2:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \beta_2 x_1^2 + 2\,\alpha_2 x_1 x_2.$$

However, the terms of higher order have been changed and may depend on Z_1 and Z_2.

To obtain a normal form of order 3 we apply our algorithm with $F_1 = A$ and

$$F_2 = D_{33} = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We compute a matrix Q such that the matrix B of Lemma 4.1 is

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \beta_3 & 3\,\alpha_3 + 3\,Z_1\,\beta_2 & 0 & 0 \end{bmatrix}.$$

Then if $\beta_2 \neq 0$ one can choose $Z_1 = -\alpha_3/\beta_2$ such that $B_{22} = 0$. Finally we obtain

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \beta_3 & 0 & 0 & 0 \end{bmatrix}.$$

And after computing the matrix representation of a diffeomorphism we obtain a rational normal form of order 3

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \beta_2 x_1^2 + 2\alpha_2 x_1 x_2 + \beta_3 x_1^3$$

which has one non-zero parameter in the homogeneous part of degree 3. We have eliminated one parameter of the homogeneous part of degree 3 in the Takens' normal form. By computations in Maple V with the above algorithm we obtain normal forms of some finite orders which we state in the following proposition.

Proposition 5.1. *Consider a dynamical system of the form* (5.1) *with matrix* (5.2) *as its linear part. Let notations be as above.*

(*a*) *If* $\beta_2 = 1$ *and* $\alpha_2 \neq 0$, *then a rational normal form of order* 15 *is*

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1^2 + 2\alpha_2 x_1 x_2 + \mu_3 x_1^3 + \mu_4 x_1^4 + \mu_5 x_1^5 + \mu_7 x_1^7 + \mu_8 x_1^8 \\ &\quad + \mu_9 x_1^{10} + \mu_{10} x_1^{11} + \mu_{11} x_1^{13} + \mu_{12} x_1^{14}, \end{aligned}$$

where, for example, $\mu_3 = \beta_3$, $\mu_4 = (-20\,\alpha_4 + 12\,\alpha_3\,\alpha_2^2 + 15\,\alpha_3\,\beta_3 + 8\,\beta_4\,\alpha_2)/(8\alpha_2)$ *and*

$$\mu_5 = \frac{40\alpha_5 - 324\beta_3\,\alpha_4 + 108\alpha_3\alpha_2^2\,\beta_3 + 243\beta_3^2\alpha_3 - 24\alpha_3\beta_4 + 56\beta_5\alpha_2}{56\alpha_2}.$$

Remark that we have eliminated all terms of degrees $6, 9, 12$ *and* 15.

(*b*) *If* $\beta_2 = 1$, $\alpha_2 = 0$ *and* $\mu_2 = \beta_3 \neq 0$, $\mu_3 = 4\alpha_4 - 3\alpha_3\beta_3 \neq 0$, $183\mu_2\mu_3 - 110\mu_4 \neq 0$, *then a rational normal form of order* 14 *is:*

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_1^2 + \beta_3 x_1^3 + \mu_3 x_1^3 x_2 + \mu_4 x_1^4 x_2 + \mu_5 x_1^6 + \mu_6 x_1^6 x_2 + \mu_7 x_1^7 x_2\\ &\quad + \mu_8 x_1^9 x_2 + \mu_9 x_1^{10} x_2 + \mu_{10} x_1^{12} x_2 + \mu_{11} x_1^{13} x_2,\end{aligned}$$

where, for example, $\mu_4 = 5\alpha_5 - 3\alpha_3\beta_4$ *and*

$$\mu_5 = \beta_6 - \frac{133}{50}\beta_3\beta_5 + \frac{567}{125}\beta_4\beta_3^2 - \frac{153}{100}\alpha_3^2\beta_3 + \frac{6}{5}\alpha_3\alpha_4 - \frac{28}{25}\beta_4^2.$$

(*c*) *If* $\beta_2 = 0$, $\alpha_2 = 1$, $4\alpha_2^2 + 9\beta_3 \neq 0$, $\beta_3 \neq 0$, *then a non-degenerate rational normal form of order* 14 *is*

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_1 x_2 + \beta_3 x_1^3 + 3\alpha_3 x_1^2 x_2 + \beta_4 x_1^4 + \beta_5 x_1^5 + \mu_6 x_1^6 + \mu_7 x_1^7 + \mu_8 x_1^8\\ &\quad + \mu_9 x_1^9 + \mu_{10} x_1^{10} + \mu_{11} x_1^{11} + \mu_{12} x_1^{12} + \mu_{13} x_1^{13} + \mu_{14} x_1^{14}.\end{aligned}$$

(*d*) *If* $\beta_2 = 0, \alpha_2 = 0$ *and* $\alpha_3\beta_3 \neq 0$, *then a non-degenerate rational normal form of order* 14 *is*

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \beta_3 x_1^3 + 3\alpha_3 x_1^2 x_2 + \beta_4 x_1^4 + \beta_5 x_1^5 + \mu_5 x_1^4 x_2 + \mu_6 x_1^6 + \mu_7 x_1^7 + \mu_8 x_1^8\\ &\quad + \mu_9 x_1^9 + \mu_{10} x_1^{10} + \mu_{11} x_1^{11} + \mu_{12} x_1^{12} + \mu_{13} x_1^{13} + \mu_{14} x_1^{14},\end{aligned}$$

where, for example, $\mu_5 = (-4\alpha_4\beta_4 + 5\alpha_5\beta_3)/\beta_3$.

All the parameters μ_j *are rational expressions depending on the coefficients of the system, and can be given explicitly. They are different in the different cases above. The non-degenerate conditions are algebraic conditions on the coefficients of the system.*

Moreover all the above normal forms are unique with respect to near identity changes of variables up to the given orders in the sense that two normal forms are equivalent by near identity transformation if and only if all the parameters in the normal forms are equal. In particular, two normal forms in two different cases are not equivalent.

One can continue to discuss other degenerate cases.

Degree	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
Takens	2	2	2	2	2	2	2	2	2
Ushiki	2	1	1						
Case (a)	2	1	1	1	0	1	1	0	1
Case (b)	1	1	1	1	1	1	1	0	1
Case (c)	1	2	1	1	1	1	1	1	1
Case (d)	0	2	1	2	1	1	1	1	1

We give, in the above table, a comparison of the normal forms derived via Takens' method and Ushiki's method up to order 10. In Ushiki (1984), Ushiki obtained a normal form of order 4. Since the goal for obtaining normal forms of dynamical systems is to eliminate as many monomials from each order as possible, we have listed in the above table the number of monomials of each degree that is still present in the normal form. For example 0 means that all terms of a given degree are eliminated (see also Chua and Kokubu, 1989, and Chen and Della Dora, 1999b). The row for the Takens' method is also valid for the methods of Cushman and Sanders (1990) and Elphick *et al.* (1987).

We also note that the work of Baider and Sanders (1992) specified nilpotent vector fields in dimension 2 into three categories. They have given unique normal forms up to any order for the first two categories. Case (c) is a particular case of the third category. Unique normal form is given in Kokubu *et al.* (1996) for case (c). An answer for the general case of the third category is given in Chen (1999).

Example 5.2. Consider dynamical systems of the form (5.1) with a zero matrix as its linear part, i.e. $A = 0$. Remark that the methods of Cushman and Sanders (1986) and Elphick *et al.* (1987) do not apply to this case. Denote

$$f_1 = \alpha_{2,0}x_1^2 + \alpha_{1,1}x_1x_2 + \alpha_{0,2}x_2^2 + \cdots, \qquad f_2 = \beta_{2,0}x_1^2 + \beta_{1,1}x_1x_2 + \beta_{0,2}x_2^2 + \cdots.$$

If $\alpha_{0,2} \neq 0$ and $\beta_{0,2} \neq 0$, then by a change of variables $x_1 = ay_1, x_2 = by_2$ with $b = 1/\beta_{0,2}, a = \alpha_{0,2}/\beta_{0,2}^2$ one reduces $\alpha_{0,2}$ and $\beta_{0,2}$ to 1. Suppose that this is done to simplify notations. We obtain a normal form of order 4.

Proposition 5.2. *Let notations be as above. If* $\alpha_{0,2} = 1$ *and* $\beta_{0,2} = 1$*, then a non-degenerate normal form of the system of order* 4 *is*

$$\begin{aligned}
\dot{x}_1 &= \alpha_{2,0}x_1^2 + \alpha_{1,1}x_1x_2 + x_2^2,\\
\dot{x}_2 &= \beta_{2,0}x_1^2 + \beta_{1,1}x_1x_2 + x_2^2 + \mu_7x_1^3 + \mu_8x_1^2x_2 + \mu_9x_1x_2^2 + \mu_{10}x_1^4 + \mu_{11}x_1^3x_2
\end{aligned}$$

where all the parameters μ_j *are rational functions on the coefficients of the system.*

One remarks that there remain 6, 3 and 2 parameters in the homogeneous part of degree 2, 3 and 4, respectively.

5.2. Examples in dimension 3

We now study dynamical systems of dimension 3 of the form

$$\dot{x} = F(x) = Ax + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}. \tag{5.4}$$

Example 5.3. Consider dynamical systems of the above form with a nilpotent matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

as its linear part.

The methods in Elphick *et al.* (1987) and Cushman and Sanders (1990) lead to the same normal form:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_3\tilde{f}_1(p_1,p_2) + x_2\tilde{f}_2(p_1,p_2) + \tilde{f}_3(p_1,p_2)$$

where $p_1 = x_1$, $p_2 = x_2^2 - 2x_1x_3$ and $\tilde{f}_1, \tilde{f}_2$ are polynomials starting at degree 1 while $\tilde{f}_3$ is a polynomial starting at degree 2. This is a general normal form but it contains as many non-zero parameters as in the Takens' normal form. We remark that p_2 is not a monomial. Note that in Chen and Della Dora (1991) we also obtained a normal form by using a Jordan basis in H_k.

Ushiki obtained further reduced normal forms of order 3 in this nilpotent case (see Ushiki, 1984, and Chua and Kokubu, 1989). In Chen and Della Dora (1999b) we obtained a further reduced normal form of order 4 by a different method.

To obtain a further reduced normal form, we first apply our algorithm to reduce the system to a new one in which $f_1 = f_2 = 0$ (see also Chen and Della Dora, 1999a). To simplify notations we shall suppose this step has been performed. Write $f_3 = \sum_{|q|\geq 2} c_q x^q$. Let

$$P = \begin{pmatrix} u & v & w \\ 0 & u & v \\ 0 & 0 & u \end{pmatrix}$$

with $u \neq 0$, then $P^{-1}AP = A$. Thus if $c_{2,0,0} \neq 0$, then one can choose $u = 1/c_{2,0,0}$ to reduce it to 1.

We obtain the following non-degenerate normal forms.

Proposition 5.3. *Let notations be as above.*

(*a*) *If* $c_{2,0,0} = 1$, *then a non-degenerate rational normal form of order* 5 *is*

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \\ &\dot{x}_3 = x_1^2 + c_{1,1,0}x_1x_2 + c_{1,0,1}x_1x_3 + c_{3,0,0}x_1^3 + \mu_5\, x_1^2x_2 + \mu_6 x_1^2x_3 \\ &\qquad +\mu_7 x_1^4 + \mu_8 x_1^3x_2 + \mu_9 x_1^5 + \mu_{10}x_1^4x_2 + \mu_{11}x_1^4x_3, \end{aligned}$$

where, for example,

$$\mu_5 = -\frac{4}{3}c_{0,0,2} + c_{2,1,0} - c_{0,2,0}c_{1,0,1} + \frac{2}{3}c_{0,2,0}^2 + \frac{1}{6}c_{0,1,1}c_{1,1,0} - \frac{1}{6}c_{0,2,0}c_{1,1,0}^2.$$

(*b*) *If* $c_{2,0,0} = 0$ *and* $c_{1,1,0} = 1$, *then a non-degenerate rational normal form of order* 5 *is*

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \\ &\dot{x}_3 = x_1x_2 + c_{1,0,1}x_1x_3 + \mu_3 x_2^2 + \mu_4\, x_1^3 + \mu_5 x_1^2x_3 + \mu_6 x_1x_2^2 + \mu_7 x_1^4 + \mu_8 x_1^3x_2 \\ &\qquad +\mu_9 x_1^3x_3 + \mu_{10}x_1^2x_2^2 + \mu_{11}x_1^5 + \mu_{12}x_1^4x_2 + \mu_{13}x_1^4x_3 + \mu_{14}x_1^3x_2^2. \end{aligned}$$

(*c*) *If* $c_{2,0,0} = 0$, $c_{1,1,0} = 0$, $c_{1,0,1} = 1$ *and* $c_{3,0,0} \neq 0$, *then a non-degenerate rational normal form of order* 4 *is*

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \\ &\dot{x}_3 = x_1x_3 + c_{0,2,0}x_2^2 + \mu_3 x_1^3 + \mu_4 x_1^2x_2 + \mu_5 x_1^2x_3 + \mu_6 x_1x_2x_3 \\ &\qquad +\mu_7 x_1^4 + \mu_8 x_1^3x_2 + \mu_9 x_1^3x_3. \end{aligned}$$

Here the parameters μ_j are as in Proposition 5.1 *and the normal forms are unique in the same sense as in Proposition* 5.1.

The following table gives a comparison for the number of non-zero parameters remaining in the non-degenerate normal forms (see also Chua and Kokubu, 1989, and Chen and Della Dora, 1999b). The row for Takens' method is also valid for the method of Elphick *et al.* (1987) and Cushman and Sanders (1990). Ushiki obtained a normal form of order 3.

Degree	2nd	3rd	4th	5th
Takens	4	6	7	9
Ushiki	3	3		
Case (a)	3	3	2	3
Case (b)	3	3	4	4

Example 5.4. Consider dynamical systems of the form (5.4) with the following nilpotent matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as its linear part. First one can apply the above algorithm to reduce the system to a new one in which $f_1 = 0$ (Theorem 4.1). Write

$$f_2 = \sum_{|q|\geq 2} b_q x^q, \qquad f_3 = \sum_{|q|\geq 2} c_q x^q.$$

We then apply the above algorithm to obtain a new system up to order 2 as follows which is in a classical normal form of order 2:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \mu_1 x_1^2 + \mu_2 x_1 x_2 + \mu_3 x_1 x_3 + \mu_4 x_2 x_3 + \mu_5 x_3^2, \\ \dot{x}_3 &= \mu_6 x_1^2 + \mu_7 x_1 x_3 + \mu_8 x_3^2, \end{aligned} \tag{5.5}$$

where $\mu_1 = b_{2,0,0}$, $\mu_2 = b_{1,1,0}$, etc. Let

$$P = \begin{pmatrix} u & v & w \\ 0 & u & 0 \\ 0 & s & r \end{pmatrix}$$

with $u, v, w, s, r \in K$ such that P is invertible. Then $P^{-1}AP = A$. The linear transformation $x \to Px$ does not change the linear part of the system. After making the linear transformation on the original system we again apply our algorithm. If $b_{2,0,0} \neq 0$, then with $u = 1/b_{2,0,0}$ one reduces $b_{2,0,0}$ to 1. In the non-degenerate case where $b_{2,0,0} = 1$ we obtain a new system of the same form as (5.5) but in which $\mu_3 = 2w + b_{1,0,1}r$, $\mu_6 = (s - c_{2,0,0})/r$. With $w = -b_{1,0,1}r/2$ and $s = c_{2,0,0}$ one obtains a new system which we write with the same notations μ_j to represent different values:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1^2 + \mu_2 x_1 x_2 + \mu_3 x_2 x_3 + \mu_4 x_3^2, \end{aligned}$$

$$\dot{x}_3 = \mu_5 x_1 x_3 + \mu_6 x_3^2,$$

where $\mu_3 = \alpha r$ and $\mu_6 = \beta r$, α and β are polynomials in the coefficients of the original system. Therefore one can reduce μ_3 or μ_6 to 1 if α or β is different from zero. We have taken $r = 1$ in our computations to simplify discussions.

Proposition 5.4. *Let notations be as above.*

(*a*) *If* $b_{2,0,0} = 1$, *and* $\mu_4 \neq 0$, *then a non-degenerate rational normal form of order* 4 *is*

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= x_1^2 + \mu_2 x_1 x_2 + \mu_3 x_2 x_3 + \mu_4 x_3^2 + \mu_7 x_1^3 + \mu_8 x_1^2 x_3 + \mu_{13} x_1^4\\
\dot{x}_3 &= \mu_5 x_1 x_3 + \mu_6 x_3^2 + \mu_9 x_1^3 + \mu_{10} x_1^2 x_3 + \mu_{11} x_1 x_3^2\\
&\quad + \mu_{12} x_3^3 + \mu_{14} x_1^4 + \mu_{15} x_1^3 x_3 + \mu_{16} x_1^2 x_3^2 + \mu_{17} x_1 x_3^3 + \mu_{18} x_3^4,
\end{aligned}$$

where, for example, $\mu_2 = b_{1,0,1} c_{2,0,0} + b_{1,1,0}$ *and* $\mu_4 = -0\frac{1}{4} b_{1,0,1}^2 + b_{0,0,2}$.

(*b*) *If* $b_{2,0,0} = 0$, $b_{1,0,1} = 1$ *and* $\mu_5 \neq 0$, *then a non-degenerate rational normal form of order* 4 *is*

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= x_1 x_3 + \mu_2 x_2 x_3 + \mu_5 x_1^3 + \mu_1 {x_1}^2 + \mu_8 x_1^2 x_3 + \mu_{12} x_1^4 + \mu_{13} x_1^3 x_2,\\
\dot{x}_3 &= \mu_3 x_1^2 + \mu_4 x_1 x_3 + \mu_5 x_3^2 + \mu_9 x_1^3 + \mu_{10} x_1^2 x_3 + \mu_{11} x_3^3 + \mu_{14} x_1^4\\
&\quad + \mu_{15} x_1^3 x_3 + \mu_{16} x_1^2 x_3^2 + \mu_{17} x_3^4
\end{aligned}$$

where $\mu_5 = b_{0,0,2}^2 c_{2,0,0} + c_{0,0,2} - b_{0,0,2} c_{1,0,1}$.

(*c*) *If* $b_{2,0,0} = 0$, $b_{1,0,1} = 0$, $c_{2,0,0} = 1$ *and* $b_{0,0,2} \neq 0$, *then a non-degenerate rational normal form of order* 4 *is*

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= \mu_1 x_3^2 + \mu_5 x_1^3 + \mu_6 x_1^2 x_3 + \mu_7 x_1 x_3^2 + \mu_{11} x_1^4 + \mu_{12} x_1^3 x_2 + \mu_{13} x_1^3 x_3,\\
\dot{x}_3 &= x_1^2 + \mu_3 x_1 x_3 + \mu_4 x_3^2 + \mu_8 x_1^3 + \mu_9 x_1^2 x_3 + \mu_{10} x_1 x_3^2 + \mu_{14} x_1^4\\
&\quad + \mu_{15} x_1^3 x_3 + \mu_{16} x_1^2 x_3^2.
\end{aligned}$$

Here the parameters μ_j *are as in Proposition* 5.1 *and the normal forms are unique in the same sense as in Proposition* 5.1.

5.3. Examples in dimension 4

We now study dynamical systems of dimension 4 of the form

$$\dot{x} = F(x) = Ax + \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}. \tag{5.6}$$

Example 5.5. First consider dynamical systems of the form (5.6) with the nilpotent matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

as the linear part. By Theorem 4.1 one first reduces the system to a new one in which $f_1 = f_2 = f_3 = 0$. We assume that this is done to simplify computations. Write

$$f_4 = \sum_{|q| \geq 2} a_q x^q.$$

Let

$$P = \begin{pmatrix} u & v & w & z \\ 0 & u & v & w \\ 0 & 0 & u & v \\ 0 & 0 & 0 & u \end{pmatrix}$$

with $u \neq 0$. Then P is invertible and $P^{-1}AP = A$. We can choose u to reduce one of the non-zero elements of degree 2 to 1. By taking $v = 0$ and choosing w, z appropriately one eliminates two more parameters in the homogeneous part of degree 2 in the normal form.

Proposition 5.5. *Let notations be as above.*

(*a*) *If* $a_{2,0,0,0} = 1$ *and* $\mu_3 = a_{1,0,1,0} \neq 0$, *then a non-degenerate rational normal form of order* 3 *is*

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_4, \\ &\dot{x}_4 = x_1^2 + \mu_2 x_1 x_2 + \mu_3 x_1 x_3 + \mu_4 x_1 x_4 + \mu_5 x_2 x_4 + \mu_6 x_1^3 \\ &\qquad + \mu_7 x_1^2 x_2 + \mu_8 x_1^2 x_3 + \mu_9 x_1^2 x_4 + \mu_{10} x_1 x_2^2. \end{aligned}$$

(*b*) *If* $a_{2,0,0,0} = 0$ *and* $a_{1,1,0,0} = 1$, *then a non-degenerate rational normal form of order* 3 *is*

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_4, \\ &\dot{x}_4 = x_1 x_2 + \mu_2 x_1 x_3 + \mu_3 x_1 x_4 + \mu_4 x_2^2 + \mu_5 x_1^3 + \mu_6 x_1^2 x_2 + \mu_7 x_1^2 x_3 + \mu_8 x_1^2 x_4 \\ &\qquad + \mu_9 x_1 x_2^2 + \mu_{10} x_1 x_2 x_3 + \mu_{11} x_1 x_2 x_4. \end{aligned}$$

(*c*) *If* $a_{2,0,0,0} = 0$, $a_{1,1,0,0} = 0$, $a_{1,0,1,0} = 1$ *and* $3 - 4a_{0,2,0,0} \neq 0$, *then a non-degenerate rational normal form of order* 3 *is*

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_4, \\ &\dot{x}_4 = x_1 x_3 + \mu_2 x_1 x_4 + \mu_3 x_2^2 + \mu_4 x_2 x_3 + \mu_5 x_1^3 + \mu_6 x_1^2 x_2 + \mu_7 x_1^2 x_3 + \mu_8 x_1^2 x_4 \\ &\qquad + \mu_9 x_1 x_2^2 + \mu_{10} x_1 x_2 x_3 + \mu_{11} x_1 x_2 x_4. \end{aligned}$$

Here the parameters μ_j *are as in Proposition* 5.1 *and the normal forms are unique in the same sense as in Proposition* 5.1.

The following table shows the numbers of non-zero elements remaining in the normal forms.

Degree	2nd	3rd
Cushman *et al.*	7	12
Case (a)	5	5
Case (b)	4	7

Example 5.6. Consider dynamical systems of the form (5.6) with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

as the linear part.

By Theorem 4.1 one can reduce f_1 and f_3 to zero. Write

$$f_2 = \sum_{|q|\geq 2} a_q x^q, \qquad f_4 = \sum_{|q|\geq 2} b_q x^q.$$

Let

$$P = \begin{pmatrix} u & v & w & z \\ 0 & u & 0 & w \\ u' & v' & r & s \\ 0 & u' & 0 & r \end{pmatrix}$$

with coefficients in K such that P is invertible. Then $P^{-1}AP = A$. We take $u' = v' = 0$ to simplify computations. By appropriately choosing u and r one reduces two non-zero parameters to 1; and by appropriately choosing w, s and z we reduce three more parameters of degree 2 to 0.

Proposition 5.6. *Let notations be as above.*

(*a*) *If* $b_{2,0,0,0} = 1$ *and* $1/u = 2a_{2,0,0,0} + b_{1,0,1,0} \neq 0$, *then a non-degenerate rational normal form of order* 3 *is*

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= \mu_1 x_1 x_3 + \mu_2 x_1 x_4 + \mu_3 x_2 x_3 + \mu_4 x_3^2 + \mu_5 x_3 x_4 + \mu_{12} x_1^3 + \mu_{13} x_1^2 x_3\\
&\quad + \mu_{14} x_1 x_3^2 + \mu_{15} x_3^3,\\
\dot{x}_3 &= x_4,\\
\dot{x}_4 &= x_1^2 + x_1 x_3 + \mu_8 x_1 x_4 + \mu_9 x_2 x_3 + \mu_{10} x_3^2 + \mu_{11} x_3 x_4 + \mu_{16} x_3^3 + \mu_{17} x_1^2 x_2\\
&\quad + \mu_{18} x_1^2 x_3 + \mu_{19} x_1^2 x_4 + \mu_{20} x_1 x_2 x_3 + \mu_{21} x_1 x_3^2 + \mu_{22} x_1 x_3 x_4 + \mu_{23} x_1 x_4^2 + \mu_{24} x_3^3,
\end{aligned}$$

where, for example, $\mu_1 = u^2(a_{1,0,1,0} - a_{2,0,0,0} b_{1,0,1,0})$, $\mu_8 = u(a_{1,1,0,0} + b_{1,0,0,1})$.

(*b*) *If* $b_{2,0,0,0} = 0$ *and* $b_{1,0,1,0} = 1$, *then a non-degenerate rational normal form of order* 3 *is*

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= \mu_1 x_1^2 + \mu_2 x_1 x_2 + \mu_3 x_1 x_3 + \mu_4 x_1 x_4 + \mu_5 x_3^2 + \mu_6 x_3 x_4 + \mu_{11} x_1^3\\
&\quad + \mu_{12} x_1^2 x_3 + \mu_{13} x_1 x_3^2 + \mu_{14} x_3^3,
\end{aligned}$$

$$\begin{aligned}
\dot{x}_3 &= x_4,\\
\dot{x}_4 &= x_1x_2 + x_1x_3 + \mu_9 x_3^2 + \mu_{10}x_3x_4 + \mu_{15}x_1^3 + \mu_{16}x_1^2x_2 + \mu_{17}x_1^2x_3\\
&\quad + \mu_{18}x_1^2x_4 + \mu_{19}x_1x_2x_3 + \mu_{20}x_1x_3x_4 + \mu_{21}x_1x_4^2 + \mu_{22}x_3^3.
\end{aligned}$$

Here the parameters μ_j are as in Proposition 5.1 *and the normal forms are unique in the same sense as in Proposition* 5.1.

The normal form of Elphick *et al.* (1987), and similarly of Cushman and Sanders (1990), is

$$\begin{aligned}
\dot{x}_1 &= x_2, & \dot{x}_2 &= x_2P_1(x_1,x_3,x_2x_3-x_1x_4) + x_4P_2(x_1,x_3,x_2x_3-x_1x_4) + Q_1(x_1,x_3),\\
\dot{x}_3 &= x_4, & \dot{x}_4 &= x_2P_3(x_1,x_3,x_2x_3-x_1x_4) + x_4P_4(x_1,x_3,x_2x_3-x_1x_4) + Q_2(x_1,x_3),
\end{aligned}$$

where P_j, Q_i are polynomials in their arguments, P_j starting at degree 1, and Q_i starting at degree 2.

The following table shows the numbers of non-zero parameters remaining in the normal forms.

Degree	2nd	3rd
Elphick *et al.*	14	24
Case (a)	11	13
Case (b)	10	12

Example 5.7. Consider dynamical systems of the form (5.6) with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & -2 & 0 \end{pmatrix}$$

as the linear part. This is a non-semi-simple matrix with $\pm i$ as its eigenvalues. By Theorem 4.1 one first reduces the system to a new one in which $f_1 = f_2 = f_3 = 0$. We assume this is done to simplify computations. We first reduce the homogeneous part of degree 2 of the system to 0. Thus assume without loss of generality that

$$f_4 = \sum_{|q|\geq 3} a_q x^q.$$

We obtain the following normal forms.

Proposition 5.7. *Let notations be as above. Then a non-degenerate rational normal form of order* 5 *is*

$$\begin{aligned}
&\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = x_4,\\
&\dot{x}_4 = -x_1 - 2x_2 + \mu_1x_1^3 + \mu_2x_1^2x_2 + \mu_3x_1^2x_3 + \mu_4x_1^2x_4 + \mu_5x_1x_2^2 + \mu_6x_1x_2x_3\\
&\qquad + \mu_7x_1x_3^2 + \mu_8x_1x_3x_4 + \mu_9x_1^5 + \mu_{10}x_1^4x_2 + \mu_{11}x_1^4x_3 + \mu_{12}x_1^4x_4 + \mu_{13}x_1^3x_2x_3,
\end{aligned}$$

where, for example, $\mu_1 = -a_{0,0,1,2}/2 + a_{0,1,1,1}/3 + a_{1,1,0,1}/3 - a_{0,0,3,0}$ *and* $\mu_2 = -2a_{0,1,0,2} - a_{0,1,2,0} + 2a_{0,2,0,1} + a_{2,1,0,0}$.

Here the parameters μ_j *are as in Proposition* 5.1 *and the normal forms are unique in the same sense as in Proposition* 5.1.

The following table shows the numbers of non-zero elements remaining in the normal forms.

Degree	2nd	3rd	4th	5th
Takens	0	8	0	12
Our method	0	8	0	5

EXAMPLE 5.8. Consider dynamical systems of the form (5.6) with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

as the linear part.

As above one can first reduce the system to a new one in which $f_1 = f_3 = 0$ by Theorem 4.1. So we can write

$$f_2 = \sum_{|q|\geq 2} a_q x^q, \qquad f_4 = \sum_{|q|\geq 2} b_q x^q.$$

PROPOSITION 5.8. *Let notations be as above.*

(*a*) *If* $b_{0,0,2,0} = 1$ *and* $a_{0,1,1,0} \neq 0$, *then a non-degenerate rational normal form of order* 4 *is*

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + \mu_1 x_1 x_3 + \mu_2 x_2 x_3 + \mu_6 x_1^3 + \mu_{10} x_1 x_3^3, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= \mu_3 x_1^2 + x_3^2 + \mu_5 x_3 x_4 + \mu_7 x_1^2 x_3 + \mu_8 x_1^2 x_4 + \mu_9 x_3^3 + \mu_{11} x_1^4 + \mu_{12} x_1^2 x_3^2 \\ &\quad + \mu_{13} x_3^4 + \mu_{14} x_3^3 x_4, \end{aligned}$$

where, for example, $\mu_1 = a_{1,0,1,0}$, $\mu_2 = a_{0,1,1,0}$, $\mu_3 = b_{0,2,0,0} + b_{2,0,0,0}$.

(*b*) *If* $b_{0,0,2,0} = 0$, $b_{0,0,1,1} = 1$ *and* $b_{0,2,0,0} + b_{2,0,0,0} \neq 0$, *then a non-degenerate rational normal form of order* 4 *is*

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + \mu_1 x_1 x_3 + \mu_2 x_2 x_3 + \mu_5 x_1 x_3^2 + \mu_6 x_2 x_3^2 + \mu_{10} x_1 x_3^3, \\ \dot{x}_3 &= x_4, \end{aligned}$$

$$\dot{x}_4 = \mu_3 x_1^2 + x_3 x_4 + \mu_7 x_1^2 x_3 + \mu_8 x_3^2 x_4 + \mu_9 x_1 x_3^2 + \mu 11 x_1^2 x_3^2$$
$$+\mu_{12} x_3^4 + \mu_{13} x_3^3 x_4.$$

Here the parameters μ_j are as in Proposition 5.1 *and the normal forms are unique in the same sense as in Proposition* 5.1.

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Originally Received 14 May 1999
Accepted 8 November 1999